Q.No→ Show that the space $R^n$ is reflexive.

or, Q.No→ The dual space of $R^n$ is isomorphic to $R^n$.

✓ In short, the dual space of $R^n$ is $R^n$.

Verification :- Let $e_1 = (1, 0, 0, \text{-----}, 0)$, $e_2 = (0, 1, 0, \text{---}, 0)$, $\text{----}$ $e_n = (0, 0, 0 \text{----}, 1)$ be a basis of $R^n$. Then any element $x = (\alpha_1, \alpha_2, \text{----}, \alpha_n) \in R^n$ can be written as,

$$x = \sum_{K=1}^{n} \alpha_K e_K$$

If $f$ is a continuous linear functional on $R^n$, then

$$f(x) = f\left(\sum_{K=1}^{n} \alpha_K e_K\right) = \sum_{K=1}^{n} \alpha_K f(e_K) = \sum_{K=1}^{n} \alpha_K B_K \quad \text{where } B_K = f(e_K)$$

Conversely, every $n$-tuple $(B_1, \text{---}, B_n) \in R^n$ determines a continuous linear function $f$ of $R^n$, given by $f(x) = \sum_{K=1}^{n} \alpha_K B_K$, where $x = (\alpha_1, \text{---} \alpha_n) \in R^n$.

By Cauchy-Schwarz inequality, we have

$$|f(x)| \leq \sum_{K=1}^{n} |\alpha_K| \, |B_K| \leq \left(\sum_{K=1}^{n} \alpha_K^2\right)^{\frac{1}{2}} \cdot \left(\sum_{K=1}^{n} B_K^2\right)^{\frac{1}{2}}$$

$$= \|x\| \left(\sum_{K=1}^{n} B_K^2\right)^{\frac{1}{2}}$$

Hence, $f$ is Continuous and

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \leq \left(\sum_{K=1}^{n} B_K^2\right)^{1/2}$$

However, if $x = (B_1, \cdots B_n)$, then

$$f(x) = \sum_{K=1}^{n} B_K^2 \text{ and } \frac{|f(x)|}{\|x\|} = \left(\sum_{K=1}^{n} B_K^2\right)^{1/2}$$

$$\therefore \|f\| = \left(\sum_{K=1}^{n} B_K^2\right)^{1/2} = \|y\|$$ where, $y = (B_1, B_2, \cdots B_n)$ $\in R^n$. Hence, the mapping from the dual space $(R^n)^*$ onto $R^n$ defined by

$$f \to y = (B_1, \cdots, B_n).$$

is norm Preserving. Clearly, the mapping is linear, one-one and onto. Therefore, it is an isomorphism. Hence the dual space or the Conjugate space of $R^n$ is $R^n$ i.e. $(R^n)^* = R^n$.

~~Hence~~, Thus $R^n$ is reflexive

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QNo→ Discuss the Conjugate space $l_p^*$ of the Space $l_p$ for $1 < p < \infty$.

95, 98, QNo→ Prove that space $l_p$ $(1 < p < \infty)$ is reflexive

or, QNo- Prove that the dual space (conjugate space) of $l_p$ is $l_q$ where $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Soln. Let $(e_K)$ be a Schauder basis for $l_p$ then every $x = (\alpha_1, \alpha_2 \cdots) \in l_p$ Can be represented uniquely as,

$$x = \sum_{K=1}^{\infty} \alpha_K e_K$$

If $f$ is a Continuous linear functional on $l_p$ i.e. $f \in l_p^*$ then,

$$f(x) = f\left(\lim_{n\to\infty} \sum_{k=1}^{n} \alpha_k e_k\right) = \lim_{n\to\infty} f\left(\sum_{k=1}^{n} \alpha_k e_k\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_k f(e_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \alpha_k \beta_k.$$

$$= \sum_{k=1}^{\infty} \alpha_k \beta_k \quad \text{where, } \beta_k = f(e_k) \text{ --- } (1)$$

We now Construct a sequence, $x_n = (\alpha_k^{(n)}) \in l_p$ by definition.

$$\left.\begin{aligned} \alpha_k^{(n)} &= \frac{|\beta_k|^q}{\beta_k} \text{ if } k \leq n \text{ \& } \beta_k \neq 0 \\ &= 0 \text{ if } k > n \text{ or } \beta_k = 0 \end{aligned}\right\} \text{ --- } (2)$$

Substituting in (1), we obtain

$$f(x_n) = \sum_{k=1}^{\infty} \alpha_k^{(n)} \beta_k = \sum_{k=1}^{n} |\beta_k|^q \text{ --- } (3)$$

By (2) and from $(q-1)p = q$, we have

$$f(x_n) \leq \|f\| \cdot \|x_n\| = \|f\| \left(\sum_{k=1}^{n} |\alpha_k^{(n)}|^p\right)^{1/p}$$

$$= \|f\| \cdot \left(\sum_{k=1}^{n} |\beta_k|^{(q-1)p}\right)^{1/p} = \|f\| \left(\sum_{k=1}^{n} |\beta_k|^q\right)^{1/p}$$

From (3),

$$\sum_{k=1}^{n} |\beta_k|^q = f(x_n) \leq \|f\| \left(\sum_{k=1}^{n} |\beta_k|^q\right)^{1/p}$$

Dividing both sides by $\left(\sum_{K=1}^{m}|\beta_K|^q\right)^{1/p}$, we have

$$\left(\sum_{K=1}^{m}|\beta_K|^q\right)^{1-1/p} = \left(\sum_{K=1}^{m}|\beta_K|^q\right)^{1/q} \leq \|f\|.$$

Since, this holds for every +ve integer $m$, letting $m \to \infty$, we obtain,

$$\left(\sum_{K=1}^{\infty}|\beta_K|^q\right)^{1/q} \leq \|f\| \quad \text{——— (4)}$$

This shows that $(\beta_K) \in l_q$. On the other hand, for every $(\beta_K) \in l_q$ we can define a continuous linear functional $g$ on $l_p$ by

$$g(x) = \sum_{K=1}^{\infty} \alpha_K \beta_K \quad \text{when } x = (\alpha_K) \in l_p.$$

The series on the R.H.S. is convergent by Holder's inequality for sums because,

$$\sum_{K=1}^{\infty}|\alpha_K \beta_K| \leq \left(\sum_{K=1}^{\infty}|\alpha_K|^p\right)^{1/p}\left(\sum_{K=1}^{\infty}|\beta_K|^q\right)^{1/q} \quad \text{——— (5)}$$

and $(\alpha_K) \in l_p$, $(\beta_K) \in l_q$.

It can be seen that $g$ is linear and continuity of $g$ follows from (5) since,

$$|g(x)| \leq \sum_{K=1}^{\infty}|\alpha_K \beta_K| \leq \|x\| \cdot M.$$

where, $M = \left(\sum_{K=1}^{\infty}|\beta_K|^q\right)^{1/q}$.

Hence, $g \in l_p^*$ Therefore, there exists a one to one correspondence between the elements of $l_p^*$ & $l_q$ and this correspondence is also onto. We finally prove that the correspondence is norm preserving.

From ① and Holder's inequality for sums, we have

$$|f(x)| = \left|\sum_{k=1}^{\infty} \alpha_k \beta_k\right| \le \left(\sum_{k=1}^{\infty} |\alpha_k|^p\right)^{1/p} \left(\sum_{k=1}^{\infty} |\beta_k|^q\right)^{1/q}$$

$$= \|x\| \left(\sum_{k=1}^{\infty} |\beta_k|^q\right)^{1/q}$$

Hence, $\|f\| = \sup\{|f(x)| : \|x\| \le 1\} \le \left(\sum_{k=1}^{\infty} |\beta_k|^q\right)^{1/q}$ — ⑥

Combining ④ & ⑥, we obtain,

$$\|f\| = \left(\sum_{k=1}^{\infty} |\beta_k|^q\right)^{1/q} = \|y\|$$

Where, $y = (\beta_k) \in l_q$. Therefore, the mapping of $l_p^*$ onto $l_q$ defined by $f \to y$ is an isomorphism.

Hence, the conjugate space of $l_p$ is $l_q$. i.e. $l_p^* = l_q$. It also follows that,

$$(l_p)^{**} = ((l_p)^*)^* = (l_q)^* = l_p.$$

Where, $\frac{1}{p} + \frac{1}{q} = 1$.

$\therefore$ $l_p$ is reflexive.

**The Dual Space of the Conjugate Space of a normed linear space:-** Let E be a normed linear space over a field K (which is R or C). Then the set B(E, K) of all Continuous linear functionals on E is a Banach space with respect to Pointwise linear operations and the norm defined by.

$$\|T\| = \sup_{\substack{x \in E \\ x \neq 0}} \frac{|T(x)|}{\|x\|}, \quad T \in B(E, K)$$

The set B(E, K) is denoted by $E^*$ and is called the (topological) dual space or the Conjugate space or the adjoint space of E. (Every element of $E^*$ is a Continuous linear functional on E). Thus the dual space of every normed linear space is a Banach space.

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Q.No> Define Conjugate of an operator. If $T^*$ is Conjugate of T then Prove that $T^*$ is Continuous and $\|T^*\| = \|T\|$.

Ans.> (Defn) **Conjugate of an operator:-** Let E be a normed space over a field K then the set $B(E, E) \equiv B(E)$ of all Continuous linear transformation of E into itself is a normed linear space with respect to Pointwise linear operations and the norm defined by

$$\|T\| = \sup_{\substack{x \in E \\ x \neq 0}} \frac{\|T(x)\|}{\|x\|}, \quad T \in B(E).$$

If E is ~~[illegible]~~ a Banach space, B(E) is also a Banach space. Every element of B(E) is called conjugate an operator on E and B(E) is the set of all operators on E. Thus the set of all operators on a Banach space is a Banach space.

Other defn:- Let X & Y be normed linear spaces and let T be a continuous linear transformation from X into Y. Define $T^*: Y^* \to X^*$ as follows.

For $g \in Y^*$ & $x \in X$, set

$$T^*(g)(x) = g(T(x))$$

The linear map, $T^*$ is a called the transpose (or conjugate ~~an operator~~ or the adjoint) of T.

We show that $T^*$ is continuous and $\|T^*\| = \|T\|$.

We have $\|T^*\| = \sup\{\|T^*(g)\| : \|g\| \leq 1\}$

$= \sup\{|[T^*(g)](x)| : \|g\| \leq 1\}$

$= \sup\{|g(T(x))| : \|g\| \leq 1 \ \& \ \|x\| \leq 1\}$

$= \sup\{\|g\| \, \|T\| \, \|x\| : \|g\| \leq 1 \ \& \ \|x\| \leq 1\}$

~~[illegible]~~ $\leq \|T\|$.

Since $Y$ is a normed linear space and $T(x)$ is a non-zero element of $Y$ there exists a functional $g_0 \in Y^*$ such that,

$$g_0(T(x)) = \|T(x)\| \ \& \ \|g_0\| = 1.$$

But $g_0(T(x)) = T^*(g_0)(x)$.

Hence $\|T^*\| \geq \|T^*(g_0)\| = \sup\limits_{\|x\| \leq 1} \|T^*(g_0)(x)\|$

$$= \sup_{\|x\| \leq 1} \|g_0(T(x))\|$$

$$= \sup_{\|x\| \leq 1} \|T(x)\| = \|T\|$$

Therefore, $\|T^*\| = \|T\|$